THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050A Mathematical Analysis I (Fall 2022)
Suggested Solution of Homework 2

(1) Note that

$$x_n = (a^n + b^n)^{\frac{1}{n}} > (a^n)^{\frac{1}{n}} = a > 0$$

and

$$x_n^n - b^n = a^n.$$

For any $\epsilon > 0$, by Archimedean property, there exists $N \in \mathbb{N}$ such that $N > \frac{a}{\epsilon}$. Then for any $n > N$, we have

$$|x_n - b| = \frac{|x_n^n - b^n|}{|x_n^{n-1} + bx_n^{n-2} + \ldots + b^{n-1}|} < \frac{a^n}{na^{n-1}} = \frac{a}{n} < \epsilon.$$

(2) Since $L < 1$, then $\frac{1-L}{2} > 0$. Since $\lim_{n\to\infty} x_n^{\frac{1}{n}} = L$, there exists $N_1 \in \mathbb{N}$ such that for any $n > N_1$, $|x_n^{\frac{1}{n}} - L| < \frac{1-L}{2}$, thus $x_n^{\frac{1}{n}} < \frac{1+L}{2} < 1$. Write $K = \frac{1+L}{2}$.

Since $K < 1$, then $1 - K > 0$. By Archimedean property, there exists $N_0 \in \mathbb{N}$ such that $N_0 > \frac{1}{1-K}$. Take $a := \max\{nK^n : n = 1, \ldots, N_0\}$.

We will prove by induction that for any $n \in \mathbb{N}$, $K^n \leq \frac{a}{n}$. When $n = 1, \ldots, N_0$, $K^n = \frac{nK^n}{n} \leq \frac{a}{n}$. When $k > N_0$, suppose $K^{k-1} \leq \frac{a}{k-1}$. Then $K^k \leq \frac{aK}{k-1} \leq (1 - \frac{1}{k})\frac{a}{k-1} = \frac{a}{k}$.

For any $\epsilon > 0$, by Archimedean property, there exists $N_1 \in \mathbb{N}$ such that $N_1 > \frac{a}{\epsilon}$. Take $N = \max\{N_0, N_1\}$. Then for any $n > N$,

$$|x_n| = x_n < K^n < \frac{a}{n} < \epsilon.$$

Hence, $\lim_{n\to\infty} x_n = 0$.

(3) (a) We first prove by induction that $x_n \geq 10$ for all $n \in \mathbb{N}$.

When $n = 1$, $x_1 = 10 \geq 10$.

Suppose $x_k \geq 10$ for some $k \in \mathbb{N}$. Then $x_{k+1} = x_k + \frac{2}{x_k} \geq 10$. Hence $x_n \geq 10$ for all $n \in \mathbb{N}$.

Suppose $\{x_n\}$ is convegent. Then $\lim_{n\to\infty} x_n = L \geq 10$. By taking limits on both sides of the equation $x_{n+1} = x_n + \frac{2}{x_n}$, since $L > 0$, we have that $L = L + \frac{2}{L} > L$. Contradiction arises. Therefore $\{x_n\}$ is not convegent.

(b) We prove by induction that $x_n < 4$ and $x_{n+1} > x_n$ for all $n \in \mathbb{N}$.

When $n = 1$, $x_1 = 1 < 4$ and $x_2 = \sqrt{5} > x_1$.

Suppose $x_k < 4$ and $x_{k+1} > x_k$ for some $k \in \mathbb{N}$. Then $x_{k+1} = \sqrt{4 + x_k} < \sqrt{8} < 4$ and $x_{k+2} = \sqrt{4 + x_{k+1}} > \sqrt{4 + x_k} = x_{k+1}$. Hence $x_n < 4$ and $x_{n+1} > x_n$ for all $n \in \mathbb{N}$.

Therefore $\{x_n\}$ is monotone increasing and bounded from above. It follows that $\{x_n\}$ is convergent.

Let $\lim_{n\to\infty} x_n = L$. Then $L = \sqrt{4 + L}$. Thus $L = \frac{1+\sqrt{17}}{2}$ or $L = \frac{1-\sqrt{17}}{2}$. Since $x_n \geq 0$, $L \geq 0$. Hence, $L = \frac{1+\sqrt{17}}{2}$.

(4) Let $L = \lim_{n\to\infty}(-1)^n x_n$. Since $\{x_{2k}\}$ is a subsequence of $\{x_n\}$, $\lim_{k\to\infty} x_{2k} = L$. Since $x_{2k} \geq 0$ for any $k \in \mathbb{N}$, $L \geq 0$.Since $\{-x_{2k-1}\}$ is a subsequence of $\{x_n\}$, $\lim_{k\to\infty} -x_{2k-1} = L$. Since $-x_{2k} \leq 0$ for any $k \in \mathbb{N}$, $L \leq 0$. Therefore, $L = 0$.

For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n > N_1$, $|x_n - 0| = |(-1)^n x_n - 0| < \epsilon$. Hence, $\lim_{n\to\infty} x_n = 0$.

The conclusion does not hold if we only assume $x_n \geq -1$. For instance, if $x_n = (-1)^n$, then $(-1)^n x_n = 1$. But $\lim_{k\to\infty} x_{2k} = 1$ and $\lim_{k\to\infty} x_{2k-1} = -1$. If $\{x_n\}$ is convergent, then $\lim_{k\to\infty} x_{2k} = \lim_{k\to\infty} x_{2k-1}$. Contradiction!

(5) Suppose for any $n \in \mathbb{N}$, $x_n < s$. We will choose a subsequence $\{n_k\} \in \mathbb{N}$ such that $x_{n_k} > s - \frac{1}{k}$ by induction. Since $s = \sup\{x_n\}$, there exists $n_1 \in \mathbb{N}$ such that $x_{n_1} > s - 1$. Suppose $n_1, \ldots n_k$ has been chosen. Since for any $n \in \mathbb{N}$, $x_n < s$, then $u := \max\{x_n : n = 1, \ldots, n_k\} < s$. Thus $\sup\{x_n : n > n_k\} = s$. (For if $v := \sup\{x_n : n > n_k\} < s$, $\max\{u, v\}$ is an upper bound of $\{x_n\}$ strictly less than s.) Therefore there exists $n_{k+1} > n_k$ such that $x_{n_{k+1}} > s - \frac{1}{k+1}$.

For any $\epsilon > 0$, by Archimedean property, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. Then for any $k > N$, we have

$$|x_{n_k} - s| = s - x_{n_k} < \frac{1}{k} < \epsilon.$$

Hence, $\lim_{k\to\infty} x_{n_k} = s$.